Measure Theory with Ergodic Horizons
Lecture 5

Now $\mathcal{A}$ is a subset of the pseudo-metric space $(\mathcal{P}(X), d)$ and we let $\mathcal{M}$ denote the closure of $\mathcal{A}$ in $\mathcal{P}(X)$ with respect to d. We will show that $\mathcal{M}$ is a σ-algebra and μ^* is finitely additive on $\mathcal{M}$ (hence ctbly additive).

Claim (b). The function $\begin{matrix} A \mapsto \mu^*(A) \\ \mathcal{P}(X) \to [0,\infty] \end{matrix}$ is continuous, in fact, 1-Lipschitz: $|\mu^*(A) - \mu^*(B)| \le d(A,B)$.

Proof. Just note that $\mu^*(A) = \mu^*(A \triangle \emptyset) = d(A, \emptyset)$, so

$$|\mu^*(A) - \mu^*(B)| = |d(A,\emptyset) - d(B,\emptyset)| \le d(A,B)$$ by the triangle inequality. $\square$

Claim (c). The function $\begin{matrix} A \mapsto A^c \\ \mathcal{P}(X) \to \mathcal{P}(X) \end{matrix}$ is continuous, in fact, an isometry: $d(A^c, B^c) = d(A,B)$.

Proof. Just note that $A^c \triangle B^c = A \triangle B$. $\square$

This implies that $\mathcal{M}$ is closed under complements: if $M \in \mathcal{M}$, then $\exists (A_n) \subseteq \mathcal{A}$ s.t. $A_n \to_d A$, so by continuity of complement, $A_n^c \to A^c$, hence $A^c \in \mathcal{M}$ because $A_n^c \in \mathcal{A}$.

Claim (d). The function $\begin{matrix} (A,B) \mapsto A \cup B \\ \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X) \end{matrix}$ is continuous, in fact, 1-Lipschitz with respect to the sum metric $d + d$ on $\mathcal{P}(X) \times \mathcal{P}(X)$.

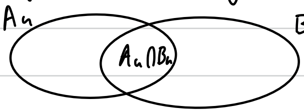
Thus, same is true for $(A,B) \mapsto A \cap B$ because it's a composition of $\cup$ and complements.

Proof. $d(A_1 \cup B_1, A_2 \cup B_2) = \mu^*((A_1 \cup B_1) \triangle (A_2 \cup B_2)) \underset{\text{mon.}}{\le} \mu^*((A_1 \triangle A_2) \cup (B_1 \triangle B_2)) \underset{\text{subadd.}}{\le} d(A_1, A_2) + d(B_1, B_2)$. $\square$

This implies that $\mathcal{M}$ is closed under finite unions: if $A, B \in \mathcal{M}$, let $A_n \to A$ and $B_n \to B$ for some $(A_n), (B_n) \subseteq \mathcal{A}$, then $A_n \cup B_n \to A \cup B$ and $A_n \cup B_n \in \mathcal{A}$, so $A \cup B \in \mathcal{M}$.
Thus, $\mathcal{M}$ is an algebra.

Claim (e). μ^* is finitely additive on the algebra $\mathcal{M}$.

Proof. Let $A,B \in \mathcal{M}$ be disjoint sets, in order to show that $\mu^*(A \sqcup B) = \mu^*(A) + \mu^*(B)$. Let $A_n \to A$ and $B_n \to B$, where $(A_n),(B_n) \subseteq \mathcal{A}$. Then by Claim (c), $\mu^*(A_n) \to \mu^*(A)$ and $\mu^*(B_n) \to \mu^*(B)$, and by Claim (d), $A_n \cup B_n \to A \sqcup B$, so again by (a), $\mu^*(A_n \cup B_n) \to \mu^*(A \sqcup B)$. But $\mu^*(A_n \cup B_n) = \mu(A_n \cup B_n) = \mu(A_n) + \mu(B_n) - \mu(A_n \cap B_n)$



$\approx_\varepsilon \mu(A_n) + \mu(B_n)$ for a fixed $\varepsilon > 0$ and large enough n, because $A_n \cap B_n \to A \cap B = \emptyset$, hence $\mu^*(A_n \cap B_n) \to \mu^*(\emptyset) = 0$. Thus, $\mu^*(A \sqcup B) \approx_\varepsilon \mu^*(A_n \cup B_n) \approx_\varepsilon \mu^*(A_n) + \mu^*(B_n) \approx_\varepsilon \mu^*(A) + \mu^*(B)$ for all large enough n, hence $\mu^*(A \sqcup B) \approx_{3\varepsilon} \mu^*(A) + \mu^*(B)$, but ε is arbitrary. $\square$

Claim (f). $\mathcal{M}$ contains ctbl unions of sets in $\mathcal{A}$.

Proof. Let $(A_n) \subseteq \mathcal{A}$ and we show that $A := \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$. By disjointification, we may assume that the A_n are pairwise disjoint. It is enough to show that $\bigsqcup_{i<n} A_i \to A$ because $\bigsqcup_{i<n} A_i \in \mathcal{A}$. But $d(\bigsqcup_{i<n} A_i, A) = \mu^*(\bigsqcup_{i \geq n} A_i) \leq \sum_{i \geq n} \mu^*(A_i) \to 0$ because the series $\sum_{i \in \mathbb{N}} \mu^*(A_i)$ converges:

$$\mu(X) \overset{\text{mon}}{\geq} \mu\Big(\bigsqcup_{i<n} A_i\Big) = \sum_{i<n} \mu(A_i) \text{ for all } n \in \mathbb{N}, \text{ hence } \sum_{i \in \mathbb{N}} \mu(A_i) \leq \mu(X) < \infty. \quad \square$$

This implies that $\mathcal{M}$ is closed under ctbl unions: let (M_n) be pairwise disjoint sets in $\mathcal{M}$ and we show that $\bigsqcup_{n \in \mathbb{N}} M_n \in \mathcal{M}$. Let $A_n \in \mathcal{A}$ be such that $d(A_n, M_n) \leq \varepsilon/2^{n+1}$. Then $d(\bigcup_{n \in \mathbb{N}} A_n, \bigcup_{n \in \mathbb{N}} M_n) \leq \mu^*(\bigcup_{n \in \mathbb{N}} A_n \triangle \bigcup_{n \in \mathbb{N}} M_n) \leq \mu^*(\bigcup_{n \in \mathbb{N}} (A_n \triangle M_n)) \underset{\text{subadd}}{\leq} \sum_{n \in \mathbb{N}} d(A_n, M_n) \leq \varepsilon$.

But by Claim (f), $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$ so $\bigcup_{n \in \mathbb{N}} M_n$ is arbitrarily close to the closed set $\mathcal{M}$, and hence is in $\mathcal{M}$.

Thus, we have shown that $\mathcal{M}$ is a σ-algebra and μ^* is finitely additive (hence ctbly additive) on $\mathcal{M}$, finishing the proof. $\square$

Carathéodory's extension (uniqueness). Let μ be a premeasure on an algebra $\mathcal{A}$ on a set X. Then $\nu \leq \mu^*$ for every extension ν of μ to a measure on $\langle \mathcal{A} \rangle_\sigma$.

If μ is σ-finite, then in fact, $\nu = \mu^*$.

Proof. For the inequality $\nu \le \mu^*$, it is enough to show that for each $B \in \langle \mathcal{A} \rangle_\sigma$ and any cbl cover $(A_n) \subseteq \mathcal{A}$ of B, we have $\nu(B) \le \sum_{n\in\mathbb{N}} \mu(A_n)$. But this is by cbl subadditivity of ν:

$$\nu(B) \underset{\text{mon}}{\le} \nu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \le \sum_{n\in\mathbb{N}} \nu(A_n) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

For the equality $\nu = \mu^*$, it is enough to prove for the case $\mu(X) < \infty$, because the σ-finite case, i.e. $X = \bigsqcup_{n\in\mathbb{N}} X_n$ where $X_n \in \mathcal{A}$ and $\mu(X_n) < \infty$, reduces to the finite case by the fact that $\nu = \sum_{n\in\mathbb{N}} \nu|_{X_n}$.

Thus, suppose $\mu(X) < \infty$. Then $\mathcal{A}$ is dense in $\langle \mathcal{A} \rangle_\sigma$ because $\langle \mathcal{A} \rangle_\sigma \subseteq \mathcal{M} = \overline{\mathcal{A}}$.
But we already know that for all $A, B \in \langle \mathcal{A} \rangle_\sigma$,

$$|\nu(A) - \nu(B)| \le \nu(A \setminus B) + \nu(B \setminus A) = \nu(A \triangle B) \le \mu^*(A \triangle B) = d(A, B),$$

so the function $\begin{aligned} A &\mapsto \nu(A) \\ \langle \mathcal{A} \rangle_\sigma &\to [0, \infty) \end{aligned}$ is 1-Lipschitz, hence continuous.

Now we have two continuous functions ν and μ^* on $\langle \mathcal{A} \rangle_\sigma$ which are equal on the dense set $\mathcal{A}$, hence they must be equal on the whole $\langle \mathcal{A} \rangle_\sigma$. $\square$

Counter-example to uniqueness for non-σ-finite premeasures.

Attempt 1. Let $\mathcal{A}$ be the algebra of finite and cofinite subsets of $\mathbb{R}$ and define μ on $\mathcal{A}$ by:

$$\mu(A) := \begin{cases} 0 & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is cofinite} \end{cases}.$$

This is indeed a premeasure on $\mathcal{A}$. The outer measure μ^* on $\mathcal{P}(X)$ is then

$$\mu^*(B) = \begin{cases} 0 & \text{if } B \text{ is cbl} \\ \infty & \text{otherwise} \end{cases}.$$

But $\langle \mathcal{A} \rangle_\sigma$ is the collection of cbl and co-cbl sets, so μ^* coincides with the Lebesgue measure on $\langle \mathcal{A} \rangle_\sigma$. Hence this is not a counter-example.

Attempt 2: Let $\mathcal{A}$ be the collection of half-open intervals $[a,b)$, where $a<b$ are extended reals from $[-\infty,\infty]$. Then note that $\mathcal{A}$ is an algebra and define:

$$\mu(A) := \begin{cases} \infty & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}$$

for all $A \in \mathcal{A}$. Then for all $S \subseteq \mathbb{R}$, $\mu^*(S) = \begin{cases} \infty & \text{if } S \neq \emptyset \\ 0 & \text{if } S = \emptyset \end{cases}$.
$\langle \mathcal{A} \rangle_\sigma$ is the Borel σ-algebra of $\mathbb{R}$ and the following are two other extensions of μ to $\langle \mathcal{A} \rangle_\sigma$:

$\mu_1 :=$ counting measure on $\mathbb{R}$, i.e. $\mu_1(B) := \begin{cases} |B| & \text{if } B \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$.

$\mu_2(B) := \begin{cases} 0 & \text{if } B \text{ is ctbl} \\ \infty & \text{otherwise} \end{cases}$. Then μ^*, μ_1, μ_2 are pairwise distinct extensions of μ to $\langle \mathcal{A} \rangle_\sigma$. $\square$

Thus, having the Lebesgue and Bernoulli(p) premeasures on the algebras that generate all Borel sets in $\mathbb{R}^d$ and $2^{\mathbb{N}}$, we get measures defined on the Borel σ-algebras of these spaces.

Def. For a metric/topological space X, we let $\mathcal{B}(X)$ denote the σ-algebra of Borel subsets of X. A **Borel measure** on X is any measure defined on $\mathcal{B}(X)$.

In particular, the Lebesgue and Bernoulli(p) measures are Borel measures.